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Robust chaos in smooth unimodal maps

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Robust chaos is defined by the absence of periodic windows and coexisting attractors in some neighborhood of the parameter space. It has been conjectured that robust chaos cannot occur in smooth systems [E. Barreto, B. Hunt, and C. Grebogi, Phys. Rev. Lett. **78**, 4561 (1997); **80**, 3049 (1998)]. Contrary to this conjecture, we describe a general procedure for generating robust chaos in smooth unimodal maps.

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The stability of a period p orbit of the smooth one-dimensional map

$$x_{t+1}=\varphi(x_t,a), \tag{1}$$

is governed by

$$m=\frac{d}{dx}\varphi^p(x,a)=\varphi'(x_p,a)\varphi'(x_{p-1},a)\dots\varphi'(x_1,a), \tag{2}$$

where the derivatives are evaluated at each point of the orbit. The periodic orbit is asymptotically stable if $|m|<1$. A periodic orbit that contains a critical point c of φ, where $\varphi'(c,a)=0$, has $m=0$ and it is called a superstable periodic orbit. For smooth one-dimensional maps a periodic window is constructed around a *spine locus*, that corresponds to parameter values that give rise to superstable periodic orbits [1].

A chaotic attractor is said to be robust if, for its parameter values, there exists a neighborhood in the parameter space with no periodic attractor and the chaotic attractor is unique in that neighborhood [2]. It has been shown that robust chaos can occur in piecewise smooth systems. Also, it has been conjectured that robust chaos cannot occur in smooth systems [1,2]. More generally, for N-dimensional maps it has been conjectured that a slight variation of n parameters can destroy the chaos if $n\geqslant k$, where k is the number of positive Lyapunov exponents. It follows that robust chaos cannot occur in smooth one-dimensional systems, where $N=n=k=1$ [2].

In a previous work we have presented a counter example to the above conjecture [3]. Here, we describe a general procedure for generating robust chaos in smooth unimodal maps.

In general, a unimodal map $\varphi(x)$ maps an interval I into itself, has a single critical point c in I, ($\varphi'(c)=0$, $c\in I$), and is monotone increasing on the left of c ($\varphi'(x)>0$, $x<c$) and monotone decreasing on the right of c ($\varphi'(x)<0$, $x>c$). The key topological property of a unimodal map is that it stretches and folds the interval I into itself.

For the sake of clarity, we will assume that

(i) $\varphi(x)$ is a unimodal map of class C^3 on the interval $x\in[0,1]$; (ii) $\varphi(x)$ has an unique maximum at the critical point $c\in[0,1]$; (iii) $\varphi(0)=\varphi(1)=0$; (iv) $\varphi(x)$ has negative Schwarzian derivative, i.e.,

$$S_\varphi(x)=\frac{\varphi'''(x)}{\varphi'(x)}-\frac{3}{2}\left(\frac{\varphi''(x)}{\varphi'(x)}\right)^2<0, \tag{3}$$

whenever $\varphi'(x)\neq 0$.

A map satisfying the above conditions is called a S-unimodal map on the interval $x\in[0,1]$. Any S-unimodal map $\varphi(y)$ on the interval $y\in[a,b]$ can be represented on the interval $x\in[0,1]$ by using the change of variable $y=a+(b-a)x$.

The most studied S-unimodal map that generates *fragile chaos* is the logistic map

$$g_\mu(x)=\mu x(1-x). \tag{4}$$

Since the logistic map has an unique maximum at $c=1/2$ and negative Schwarzian derivative, there can be at most one attracting periodic orbit with the critical point in its basin of attraction, for any $\mu\in[0,4]$. When $\mu=4$ the orbit with initial value $x_0=c$ maps in two iterates to the fixed point $x=0$, which is unstable $|g_4'(0)|=4>1$. It follows that for $\mu=4$ the logistic map does not have any stable periodic orbits and there is an unique chaotic attractor [4]. However, if the parameter $\mu=4$ is slightly modified, the chaotic attractor is destroyed.

Now, let us try to find a general procedure for which a S-unimodal map $f_\nu(x)$ on the interval $x\in[0,1]$, can generate *robust chaos* in a large neighborhood of the parameter space, ν. First we give the following lemma.

Lemma. If $\varphi(x)$ is a S-unimodal map on the interval $x\in[0,1]$ with an unique maximum at the critical point $c\in[0,1]$, then

$$f_\nu^{(\pm)}(x)=\frac{1-\nu^{\pm\varphi(x)}}{1-\nu^{\pm\varphi(c)}},\quad \forall\nu>0,\ \nu\neq 1, \tag{5}$$

is also a S-unimodal map on the interval $x\in[0,1]$ with an unique maximum at the same critical point c.

Proof.

(1) We have

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$$f_\nu^{(\pm)\prime}(x)=\frac{\mp\ln(\nu)\nu^{\pm\varphi(x)}}{1-\nu^{\pm\varphi(c)}}\varphi'(x), \tag{6}$$

$$f_\nu^{(\pm)\prime\prime}(x)=\frac{\mp\ln(\nu)\nu^{\pm\varphi(x)}}{1-\nu^{\pm\varphi(c)}}[\varphi''(x)\pm\ln(\nu)(\varphi'(x))^2], \tag{7}$$

$$f_\nu^{(\pm)\prime\prime\prime}(x)=\frac{\mp\ln(\nu)\nu^{\pm\varphi(x)}}{1-\nu^{\pm\varphi(c)}}[\varphi'''(x)\pm 3\ln(\nu)\varphi'(x)\varphi''(x) + \ln^2(\nu)[\varphi'(x)]^3]. \tag{8}$$

From the above equations it follows that if $\varphi(x)$ is a unimodal map of class C^3 on the interval $x\in[0,1]$, then also $f_\nu^{(\pm)}(x)$ is a unimodal map of class C^3 on the interval $x\in[0,1]$, $\forall\nu>0,\nu\neq 1$ ($f_\nu^{(\pm)}(x)$ and its derivatives are compositions of $\varphi(x)$ and its derivatives).

(2) Let $\varphi(c)$ be the unique maximum of the S-unimodal map $\varphi(x)$ at the critical point $c\in[0,1]$. Then $\varphi(x)\leqslant\varphi(c)$, $\forall x\in[0,1]$ and obviously, $f_\nu^{(\pm)\prime}(c)=0$, $f_\nu^{(\pm)}(x)\leqslant f_\nu^{(\pm)}(c)=1$, $\forall x\in[0,1]$, $\forall\nu>0,\nu\neq 1$.

(3) $f_\nu^{(\pm)}(0)=f_\nu^{(\pm)}(1)=0$, $\forall\nu>0,\nu\neq 1$.

(4) The Schwarzian derivative is negative,

$$S_{f_\nu^{(\pm)}}(x)=S_\varphi(x)-\frac{1}{2}[\ln(\nu)\varphi'(x)]^2<0, \tag{9}$$

because $S_\varphi(x)<0$, whenever $\varphi'(x)\neq 0$, $\forall\nu>0$, $\nu\neq 1$ (end of proof).

The S-unimodal map $f_\nu^{(\pm)}(x)$ has two fixed points on the interval $x\in[0,1]$. The first fixed point is $a=0$, $f_\nu^{(\pm)}(0)=0$, $\forall\nu>0$. The second fixed point is the solution $b\in(c,1)$ of the transcendent equation $f_\nu^{(\pm)}(b)=b$, $\forall\nu>0$. Now, we can formulate the following more general theorem.

Theorem. If $\varphi(x)$ is a S-unimodal map on the interval $x\in[0,:1]$ with an unique maximum at the critical point $c\in[0,1]$, then

$$f_\nu^{(\pm)}(x)=\frac{1-\nu^{\pm\varphi(x)}}{1-\nu^{\pm\varphi(c)}} \tag{10}$$

is a S-unimodal map on the interval $x\in[0,1]$ and it generates *robust chaos* for any value of ν satisfying the condition

$$f_\nu^{(\pm)\prime}(0)=\left|\frac{\ln(\nu)\varphi'(0)}{1-\nu^{\pm\varphi(c)}}\right|>1,\ \nu>0,\ \nu\neq 1. \tag{11}$$

Proof. The proof is the same as in the case of logistic map with $\mu=4$. Obviously, according to the previous Lemma, $f_\nu^{(\pm)}(x)$ is a S-unimodal map on the interval $x\in[0,1]$, $\forall\nu>0$, $\nu\neq 1$. Since the map has an unique maximum at $x=c$ and negative Schwarzian derivative, there can be at most one attracting periodic orbit with the critical point in its basin of attraction, for any $\nu>0$, $\nu\neq 1$. The orbit with initial value $x_0=c$ maps in two iterates to the fixed point $x=0$. This fixed point $x=0$ is unstable if condition (11) is satisfied. In this case, the map does not have any stable periodic orbits and there is an unique chaotic attractor (end of proof).

Let us to consider an example for the above theorem. Here, we consider the following simple quadratic map:

$$\varphi(x)=x(1-x). \tag{12}$$

Obviously, this map is S-unimodal because

(A) $\varphi(x)$ is an unimodal map of class C^3 on the interval $x\in[0,1]$;

(B) $\varphi(x)$ has an unique maximum at the critical point $c=1/2\in[0,1]$;

(C) $\varphi(0)=\varphi(1)=0$;

(D) $\varphi(x)$ has negative Schwarzian derivative:

$$S_\varphi(x)=-\frac{3}{2}\left(\frac{-2}{1-2x}\right)^2<0,\quad \forall x\in[0,1],\quad x\neq c=1/2.$$

The map has a stable fixed point $a=0$: $\varphi(a)=0$, $\varphi'(a)=1$. Therefore, the orbit with any initial value $x_0\in[0,1]$ will end at $x=a=0$. In fact, the above map is a fixed point application.

Now let us to consider the S-unimodal map $f_\nu^{(\pm)}(x)$ given by the Eq. (10) with $\varphi(x)$ given by the Eq. (12),

$$f_\nu^{(\pm)}(x)=\frac{1-\nu^{\pm x(1-x)}}{1-\nu^{\pm 1/4}}. \tag{13}$$

Obviously, $f_\nu^{(\pm)}(x)$ is an unimodal map of class C^3 on the interval $x\in[0,1]$. Also, $f_\nu^{(\pm)}(1/2)=1$ is the unique maximum at the critical point $c=1/2$ and $f_\nu^{(\pm)}(0)=f_\nu^{(\pm)}(1)=0$, $\forall\nu>0$, $\nu\neq 1$.

The Schwarzian derivative is negative:

$$S_{f_\nu^{(\pm)}}(x)=-\frac{3}{2}\left(\frac{-2}{1-2x}\right)^2-\frac{1}{2}[\ln(\nu)(1-2x)]^2<0, \tag{14}$$

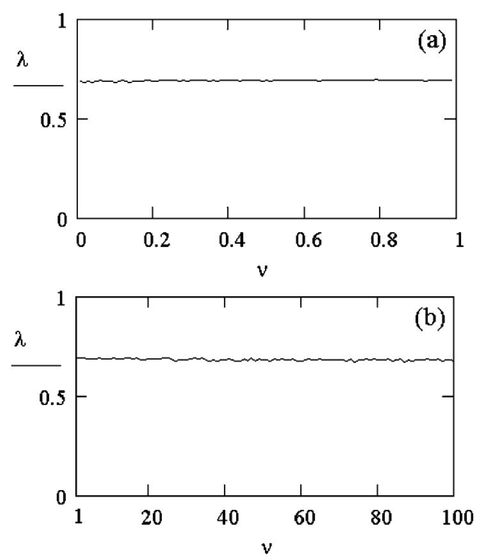

FIG. 1. The Lyapunov exponent (18) of the S-unimodal maps (16) (a) and (17) (b).

$\forall x \in [0,1], x \neq c = 1/2$, $\forall \nu > 0$, $\nu \neq 1$.

The fixed point $x=0$ is unstable for

$$f_\nu^{(\pm)\prime}(0) = \left| \frac{\ln(\nu)}{1-\nu^{\pm 1/4}} \right| > 1. \tag{15}$$

But, we have $|1-\nu^{1/4}| < |\ln(\nu)|$, $\forall \nu \in (0,1)$. Therefore, the map

$$f_\nu^{(+)}(x) = \frac{1-\nu^{x(1-x)}}{1-\nu^{1/4}} \tag{16}$$

is generating *robust chaos* for $\nu \in (0,1)$.

Also, we have $|1-\nu^{-1/4}| < |\ln(\nu)|$, $\forall \nu \in (1,\infty)$, and the map

$$f_\nu^{(-)}(x) = \frac{1-\nu^{-x(1-x)}}{1-\nu^{-1/4}} \tag{17}$$

is generating *robust chaos* for $\nu \in (1,\infty)$.

In Fig. 1 we give the Lyapunov exponent

$$\lambda(\nu) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \left| \frac{d}{dx} f_\nu^{(\pm)}(x_t) \right|, \tag{18}$$

calculated for the above maps. One can see that the Lyapunov exponent is positive, $\forall \nu > 0$, $\nu \neq 1$. Moreover, in this example, $\lambda(\nu) \cong \ln(2)$. The bifurcation diagram of these maps is a black rectangle, without any periodic windows (we don't give the figure for aesthetically reasons). The numerical computation has been performed using very long orbits, $T = 10^6$, and an accuracy of 500 digits, using an arbitrary precision mathematical library (MAPLE).

In conclusion, contrary to the conjecture that robust chaos cannot be generated by smooth maps, we have derived a *general procedure* for generating *robust chaos* in *smooth unimodal maps*.

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